Grant's Tutoring

MATRICES

for MANAGEMENT

Volume 1 of 2

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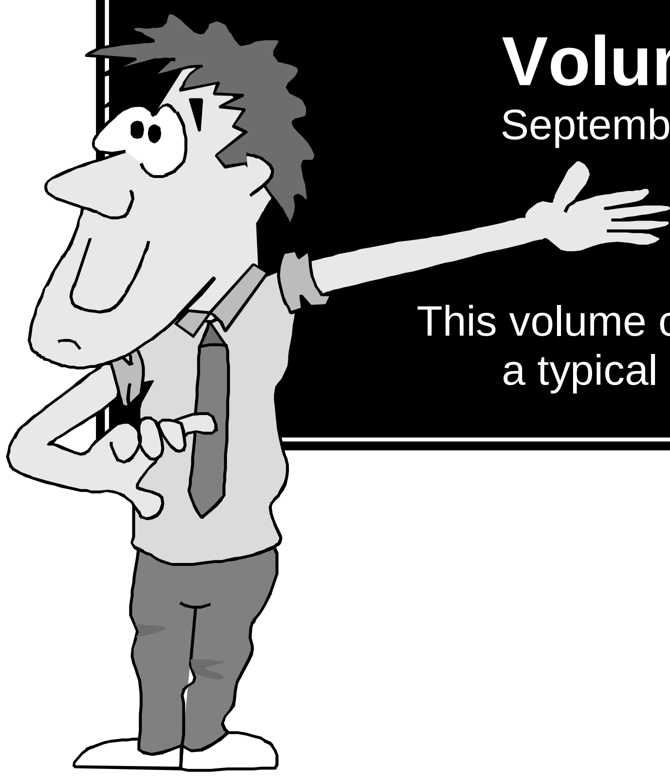

This volume covers the topics on
a typical midterm exam.

Learn What You Need to Know

Know What You Need to Learn

While studying this book, why not hear Grant explain it to you? Contact Grant for info about purchasing **Grant's Audio Lectures**. Some concepts make better sense when you hear them explained.

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HOW TO USE THIS BOOK

I have broken the course up into lessons. Study each lesson until you can do all of my lecture problems from start to finish without any help. Then do the Practise Problems for that lesson. If you are able to solve all the Practise Problems I have given you, then you should have nothing to fear about your Midterm or Final Exam.

I have presented the course in what I consider to be the most logical order. Although my books are designed to follow the course syllabus, it is possible your prof will teach the course in a different order or omit a topic. It is also possible he/she will introduce a topic I do not cover. **Make sure you are attending your class regularly! Stay current with the material, and be aware of what topics are on your exam. Never forget, it is your prof that decides what will be on the exam, so pay attention.**

Note that the Distance Ed course does the course in a very different order. Compare your outline to my Table of Contents and feel free to contact me if you are not sure what lesson to do when.

If you have any questions or difficulties while studying this book, or if you believe you have found a mistake, do not hesitate to contact me. My phone number and website are noted at the bottom of every page in this book. "Grant's Tutoring" is also in the phone book. **I welcome your input and questions.**

Wishing you much success,

Grant Skene

Owner of Grant's Tutoring

Lesson 2: Row-Reduction and Linear Systems

The Rank of a Matrix:

- The rank of a matrix equals the number of leading 1's it would have in its row-reduced echelon form.
- If a system is consistent (one or infinite solutions), the rank of the coefficient matrix is equal to the rank of the augmented matrix.
- If a system is inconsistent, the rank of the coefficient matrix is less than the rank of the augmented matrix. (The augmented matrix will have a rank that is one higher than the coefficient matrix.)
- The rank of the coefficient matrix could never be more than the rank of the augmented matrix.

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Suppose that the following matrices are the row echelon form of the augmented matrix of a system of linear equations. For each matrix answer the following questions:

(i) How many equations and how many variables were in the original system?

(ii) What is the rank of the coefficient matrix and the augmented matrix?

(iii) How many parameters are in the solution?

(iv) List the solution(s), if possible.

(a) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right)$

(b) $\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$

(c) $\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

2. Consider the system

$$\begin{aligned} 2x + 3y + z &= a \\ x + z &= b. \\ y - 2z &= c \end{aligned}$$

Suppose (1, 2, –1) is a solution to this system, find a, b and c.

3. Solve the following systems of equations using Gauss-Jordan elimination.

(a)
$$\begin{aligned} 2x_1 + 2x_2 - x_3 + x_5 &= 2 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 5 \\ x_1 + x_2 - 2x_3 - x_5 &= -2 \\ x_3 + x_4 + x_5 &= 1 \end{aligned}$$

(b)
$$\begin{aligned} 3x + 7y + 2z &= 9 \\ 2x + 4y + 2z &= 4 \\ x + 3y - z &= 4 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ 2x_1 + 3x_2 + 3x_3 &= 1 \\ -x_1 - 2x_2 - 2x_3 + x_4 &= 0 \\ - x_2 - x_3 + 2x_4 &= 1 \end{aligned}$$

(d)
$$\begin{aligned} 2x + y + z &= 2 \\ y - z &= -1 \\ x + z &= 1 \end{aligned}$$

4. Solve the system of equations

$$\begin{aligned} -y + z &= 3 \\ x - y - z &= 0 \\ -x - z &= -3 \end{aligned}$$

using Gaussian elimination and back substitution.

5. Solve the two systems of equations below simultaneously:

$$\begin{array}{rcrcrcr} x & + & 6y & + & 3z & = & 34 \\ x & + & 6y & + & 2z & = & 30 \\ & & 2y & + & 2z & = & 14 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcr} x & + & 6y & + & 3z & = & 30 \\ x & + & 6y & + & 2z & = & 24 \\ & & 2y & + & 2z & = & 16 \end{array}$$

6. Given the system of equations

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & 2x_3 & = & 0 \\ & & x_2 & - & x_3 & = & k, \\ -x_1 & + & 2x_2 & - & 3x_3 & = & 1 \end{array}$$

find, if possible, the value of k if

(a) the system has infinite solutions.

(b) the system has a unique solution.

(c) the system has no solution.

7. Given the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & a & b \end{array}\right),$$

find conditions on real numbers a and b such that:

(a) the system has no solution.

(b) the system has a unique solution.

(c) the system has infinitely many solutions.

8. A linear system of equations has been row-reduced into this augmented matrix (it is not necessarily in RREF)

$$\left(\begin{array}{ccc|c} 1 & 0 & a+1 & 7 \\ 0 & 1 & -5 & 6 \\ 0 & 0 & a^2-4a & a-4 \end{array}\right),$$

find all real numbers a such that:

(a) the system has infinitely many solutions.

(b) the system has no solution.

(c) the system has a unique solution.

9. Anne, Betty and Carol went to their local produce store to purchase some fruit. Anne bought one pound of apples and two pounds of bananas and paid \$1.85. Betty bought two pounds of apples and one pound of grapes and paid \$3.65. Carol bought one pound of bananas and two pounds of grapes and paid \$3.95. Find the price per pound for each of the three fruits.

10. A company owns three types of trucks. These trucks are equipped to haul two different types of machines per load. Truck 1 can haul 2 of machine A and 3 of machine B. Truck 2 can haul 1 of machine A and 2 of machine B. Truck 3 can haul 3 of machine A and 4 of machine B. Assuming each truck is fully loaded, how many trucks of each type should be sent to haul exactly 18 of machine A and 26 of machine B. If there is more than one possible solution provide all possible solutions, keeping in mind that the company can use no more than 4 of any particular type of truck.

11. List <u>all</u> 3×2 row-reduced echelon form matrices.

12. Consider the linear equation with three variables: $ax + by + cz = d \quad (1)$

where a, b, c, and d are any real number but $d \neq 0$.

Then, the associated homogeneous equation would be: $ax + by + cz = 0 \quad (2)$.

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two solutions to equation (1), and let (x_0, y_0, z_0) be a solution to equation (2).

(a) Show $(x_1 - x_2, y_1 - y_2, z_1 - z_2)$ is a solution to equation (2).

(b) Show $(x_1 - x_0, y_1 - y_0, z_1 - z_0)$ is a solution to equation (1).

(c) Show (kx_0, ky_0, kz_0) is a solution to equation (2) where k is any real number.

Lesson 3: Matrix Math

Important Matrix Facts and Definitions:

- **An identity matrix *I*** is a square matrix ($n \times n$) where every number on the main diagonal is 1. All other entries are 0. e.g. The 3×3 identity matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

  Multiplying a matrix with I is essentially like multiplying by 1:

$$AI = IA = A$$

- **A zero matrix 0** is a matrix of any size ($m \times n$) where every entry is 0.

  e.g. The 2×3 zero matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Multiplying a matrix with **0** is essentially like multiplying by 0, the answer is a zero matrix of appropriate size:

$$A\mathbf{0} = \mathbf{0} \text{ and } \mathbf{0}A = \mathbf{0}$$

- **The inverse of matrix *A* is denoted A^{-1}:**

$$AA^{-1} = I \text{ and } A^{-1}A = I$$

- **Matrix *A* is symmetric if $A^T = A$.** To make your own symmetric matrix, put any numbers you want in Row 1 then put those same numbers in the same order down Column 1. Now put whatever you want in Row 2 and fill the exact same numbers down Column 2. Continue until you have made the matrix of the desired size. Note that all symmetric matrices are square ($n \times n$).

- **Matrix *A* is skew-symmetric if $A^T = -A$.** To make your own skew-symmetric matrix, first, **you must have 0's down the main diagonal** (starting in the top left corner). Then put any numbers you want for the rest of Row 1 and put those same numbers but with opposite signs in the same order down Column 1. Now put whatever you want in the rest of Row 2 and fill the exact same numbers but with opposite signs down Column 2. Continue until you have made the matrix of the desired size. Note that all skew-symmetric matrices are square ($n \times n$).

Diagonal Matrices:

- ☑ A *diagonal* matrix has strictly 0's off of the main diagonal.

$$\text{e.g.} \quad A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix} \text{ is a diagonal matrix.}$$

Triangular Matrices:

- ☑ An *upper triangular* matrix has strictly 0's *below* the main diagonal.
- ☑ A *lower triangular* matrix has strictly 0's *above* the main diagonal.

$$\text{e.g.} \quad B = \begin{pmatrix} -2 & 3 & 4 & 5 \\ 0 & 6 & 1 & 7 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 13 & 2 & 0 & 0 \\ -4 & 7 & -3 & 0 \\ 8 & 10 & 6 & 10 \end{pmatrix}$$

B is upper triangular and C is lower triangular

Properties of Inverse and Transpose Matrices:

Inverse	Transpose
$\left(A^{-1}\right)^{-1} = A$	$\left(A^T\right)^T = A$
$(AB)^{-1} = B^{-1}A^{-1}$	$(AB)^T = B^T A^T$
$(kA)^{-1} = \frac{1}{k}A^{-1}$	$(kA)^T = kA^T$
$(A+B)^{-1} \neq A^{-1} + B^{-1}$	$(A+B)^T = A^T + B^T$

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$$

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Let $A = \begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 12 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 6 & -1 \\ 0 & 2 \\ -4 & -7 \end{pmatrix}$.

Do the matrix operation indicated, <u>OR</u> explain briefly why the operation is impossible.

(a) $A + B$ **(b)** $B^T + 3C$

(c) CA **(d)** BA

(e) $AB + BC$ **(f)** $A^{-1}(AB)$

(g) $A^{-1}A^t$ **(h)** A^2

(i) B^2 **(j)** $A^{-1}A^3$

2. Let $A = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. Compute the following.

(a) A^{-1} **(b)** B^{-1} **(c)** A^3

(d) B^2 **(e)** B^{-3} **(f)** A^{-2}

3. If A, B and C are matrices such that $(AB)C^T$ is defined, and the size of A is 5×4, B is square and C is $p \times q$, what size is $(AB)C^T$? Do you know what p or q must be? Explain.

4. Give an example of 2×2 matrices A, B and C, if $AB = AC$, $A \neq 0$, and $B \neq C$.

5. Let A, B and C be $n \times n$ matrices such that $2AB - 3AC = I_n$. Indicate how you can tell A^{-1} exists, and find A^{-1} in terms of B and C.

6. Let $A = [a_{ij}]$ be a 2×2 matrix. Given that $A + A^T - 4I = 0$ and $a_{21} = 3$, find A.

Lesson 4: The Inverse of a Matrix and Applications

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

(a) Find the inverse of A.

(b) Use your answer to part (a) to solve the system $A\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

2. Consider the matrices $A = \begin{pmatrix} 1 & -2 \\ -2 & c \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$.

(a) What value must c have so that the system $A\mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is inconsistent?

(b) What value must d have so that the system $B\mathbf{x} = \begin{pmatrix} 1 \\ d \end{pmatrix}$ has infinitely many solutions?

3. **(a)** Solve $\begin{pmatrix} 2 & 4 & -1 & -15 \\ -3 & -6 & 2 & 26 \\ 3 & 6 & 3 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 15 \end{pmatrix}$.

(b) Find a particular solution of (a) with $x_4 = 2$.

(c) Find a particular solution of (a) with $x_1 = 5$.

(d) Solve $\begin{pmatrix} 2 & 4 & -1 & -15 \\ -3 & -6 & 2 & 26 \\ 3 & 6 & 3 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

4. Let A and B be 3×3 matrices and $\mathbf{b} = \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$.

Given the following information: $A \xrightarrow{\text{add 3 times row 1 to row 2}} B \xrightarrow{\text{multiply row 3 by } \frac{1}{4}} I$

(a) Find A^{-1}. **(b)** Solve $A\mathbf{x} = \mathbf{b}$.

5. Let $C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}$.

(a) Compute C^{-1}.

(b) Find a matrix B, such that $CB = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

(c) Find a matrix A such that $AC = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \end{pmatrix}$.

6. Show that, if $\mathbf{s}$ and $\mathbf{t}$ are solutions to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{s} - \mathbf{t}$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

7. Let A, B and C be square matrices such that $AB = I$ and $BC = I$. Is it necessarily true that $A = C$? Justify your answer.

Lesson 5: Elementary Matrices

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. For the matrices below, determine if the matrix is elementary. If it is, state the elementary row operation it represents and state the inverse elementary matrix.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ **(c)** $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ **(e)** $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ **(f)** $\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2. In each of the following cases, find an elementary matrix E that satisfies the given equation.

(a) $E\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 6 & 7 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 1 \end{pmatrix}$

(b) $E\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 2 & 6 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

(c) $E\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 3 & 2 & 3 \\ 7 & 8 & 9 & 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 7 & 8 & 9 & 7 & 8 & 9 \\ 2 & 3 & 2 & 3 & 2 & 3 \end{pmatrix}$

Lesson 6: Determinants and Their Properties

Important Determinant Facts and Properties:

- ☑ The determinant of a *diagonal* or *triangular* matrix is simply the *product* of the main diagonal. For example, given:

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 3 & 4 & 5 \\ 0 & 6 & 1 & 7 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 13 & 2 & 0 & 0 \\ -4 & 7 & -3 & 0 \\ 8 & 10 & 6 & 10 \end{pmatrix}$$

$$|A| = -240, \quad |B| = 24, \quad |C| = -300.$$

Elementary row operations have the following effects on a determinant:

- ☑ If you interchange 2 rows (or 2 columns), you change the *sign* of a determinant.
- ☑ If you *multiply* a row (or column) by any scalar, the determinant is also multiplied by that scalar.
- ☑ If you *add* or *subtract* a multiple of one row to another row, the determinant is *unchanged*. The same is true for columns.

Which is to say, given a square matrix A where det(A) = k, if:

Row Operation or Column Operation	**Effect on det(*A*) = *k***
$R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$	new determinant = $-k$
$R_i \rightarrow cR_i$ or $C_i \rightarrow cC_i$	new determinant = ck
$R_i \rightarrow R_i \pm cR_j$ or $C_i \rightarrow C_i \pm cC_j$	new determinant = k (unchanged)

Determinant Properties:

- ☑ If A is an $n \times n$ matrix, then $|kA| = k^n |A|$.
- ☑ $|AB| = |A| \cdot |B|$
- ☑ $|A^{-1}| = \dfrac{1}{|A|}$
- ☑ $|A^T| = |A|$
- ☑ $|A^2| = |A|^2$; $|A^3| = |A|^3$;...$|A^n| = |A|^n$
- ☑ $|A+B| \neq |A| + |B|$ (there is no property for $|A+B|$).
- ☑ If matrix A has a 0-row or 0-column, then $|A| = 0$.
- ☑ If matrix A has 2 identical rows or 2 identical columns, then $|A| = 0$.
- ☑ If matrix A has one row which is merely a multiple of another row (or if one column is merely a multiple of another column), then $|A| = 0$.
- ☑ If matrices A, B and C are all identical except for the rth row (for example, maybe the 2nd row is different in all the matrices), and, if adding the rth row of A to the rth row of B actually makes the rth row of C, then:

$$|C| = |A| + |B|$$

The same property applies for columns.

(This is, without doubt, the world's most complicated property of determinants, and I wouldn't worry about trying to remember it.)

The "Equivalent Statements" Theorem:

If A is an $n \times n$ matrix, then the following statements are equivalent (i.e., If one statement is known to be true for a given matrix A, then all the statements are true; if one statement is known to be false, then all the statements are false.)

(a) $\det(A) \neq 0$.

(b) A is invertible.

(c) The reduced row-echelon form of A is I.

(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (i.e., the only solution is (0, 0, 0, ...)).

(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

(g) The rank of A is n.

(h) A is expressible as a product of elementary matrices.

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Compute the determinants of the following matrices:

(a) $A = \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix}$ **(b)** $B = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 8 & -4 \\ 2 & 3 & -5 \end{pmatrix}$ **(c)** $C = \begin{pmatrix} 2 & 6 & 3 & 11 \\ 0 & 2 & -6 & 6 \\ 0 & 1 & 3 & -3 \\ 1 & 3 & -1 & 7 \end{pmatrix}$

2. Suppose A and B are 4×4 matrices with $|A| = 3$ and $|B| = -2$. Find $|C|$ if:

(a) C is obtained from A by multiplying row 3 of A by 2.

(b) C is obtained from B by interchanging rows 2 and 4.

(c) C is obtained from A by subtracting 3 times row 3 of A from row 1 of A.

(d) $C = 2A$ **(e)** $C = -B$ **(f)** $C = A^{-1}$ **(g)** $C = AB$ **(h)** $C = B^T$

3. Let A be a 3×3 matrix and suppose $\det(A) = 5$. If B and C are obtained from A by the row operations:

$$A \xrightarrow{R_1 \to 3R_2 + R_1} B$$

$$B \xrightarrow{R_1 \to 3R_1} C$$

find $\det(B)$ and $\det(C)$.

4. If S is an invertible matrix and $B = S^{-1}AS$, prove that $|A| = |B|$.

5. Let $A = \begin{pmatrix} x & 0 & -3x \\ x & x-1 & -3 \\ 0 & 0 & 2x-1 \end{pmatrix}$.

(a) Find the determinant of A.

(b) For what values of x is A not invertible?

6. Compute the determinants of the matrices below by first row-reducing them into triangular form.

(a) $\begin{pmatrix} 3 & -8 & 11 \\ 1 & -2 & 3 \\ -2 & 8 & -16 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 & 6 & 3 & 11 \\ 0 & 2 & -6 & 6 \\ 0 & 1 & 3 & -3 \\ 1 & 3 & -1 & 7 \end{pmatrix}$

7. Let $A = \begin{pmatrix} 3 & -8 & 11 \\ 1 & -2 & 3 \\ -2 & 8 & -16 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 6 & 3 & 11 \\ 0 & 2 & -6 & 6 \\ 0 & 1 & 3 & -3 \\ 1 & 3 & -1 & 7 \end{pmatrix}$

(a) Compute the A_{31} minor. **(b)** Compute the A_{12} cofactor.

(c) Compute the B_{34} cofactor. **(d)** Compute the B_{13} cofactor.

Lesson 7: The Adjoint Matrix

Memorize this Formula:

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. If $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -3 & 1 \\ 0 & -7 & 2 \end{pmatrix}$,

(a) Find adj(A). **(b)** Find A^{-1}

(c) Solve the system
$$\begin{array}{rcrcrcr} x & + & 2y & & & = & 4 \\ 2x & - & 3y & + & z & = & -2 \\ & - & 7y & + & 2z & = & 5 \end{array}$$

2. If $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$, then $\text{adj}(A) = \begin{pmatrix} -2 & 2 & 0 & 1 \\ p & -1 & -1 & -1 \\ 0 & q & 0 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix}$. Find p and q.

3. A and B are 3 × 3 matrices where $|A| = 2$ and $|B| = 3$.

(a) Find A (adj(A)). **(b)** Find $|A\ (\text{adj}(A))|$.

(c) Find AB (adj(AB)). **(d)** Find $|AB\ (\text{adj}(AB))|$.

Lesson 8: Cramer's Rule

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. For the given systems of equations below, determine if Cramer's Rule can be used to solve the system. If the answer is "yes", solve the system.

(a)
$$\begin{aligned} x + 2y + z &= 3 \\ 3x + y - z &= -1 \\ 5x - 3z &= -5 \end{aligned}$$

(b)
$$\begin{aligned} 2x + 3y - z &= 1 \\ x + 4y + 2z &= 2 \\ 3x - y - z &= 3 \end{aligned}$$

Lesson 9: Markov Analysis

Important Facts about Markov Analysis:

- ☑ Every column in the transition matrix T will total to exactly 1. Similarly, the column of the initial state vector, $\mathbf{x}^{(0)}$, the state after one time period, $\mathbf{x}^{(1)}$, two time periods, $\mathbf{x}^{(2)}$, etc., will also total to 1.

- ☑ Given the transition matrix, T, and the initial state vector, $\mathbf{x}^{(0)}$, we can compute the state after one time period, $\mathbf{x}^{(1)}$, two time periods, $\mathbf{x}^{(2)}$, etc., by a **Markov Chain**:

$$T\mathbf{x}^{(0)} = \mathbf{x}^{(1)}$$
$$T\mathbf{x}^{(1)} = \mathbf{x}^{(2)}$$
$$T\mathbf{x}^{(2)} = \mathbf{x}^{(3)}$$

- ☑ This also means we can express the state at any time period, $\mathbf{x}^{(n)}$, in terms of T and $\mathbf{x}^{(0)}$, the initial givens:

$$T\mathbf{x}^{(0)} = \mathbf{x}^{(1)}$$
$$T^2\mathbf{x}^{(0)} = \mathbf{x}^{(2)}$$
$$T^3\mathbf{x}^{(0)} = \mathbf{x}^{(3)}$$
$$\vdots$$
$$T^n\mathbf{x}^{(0)} = \mathbf{x}^{(n)}$$

- ☑ If T has positive numbers in every position, then it is a **regular** transition matrix and we have a **regular Markov process**. If any element in T is 0 or a negative number, the process will still be regular provided some power of T has strictly positive elements. (If T^2, or T^3, or any power of T, T^n, has all positive elements, the process is regular.)

- ☑ If we have a **regular** Markov process, the process will reach equilibrium. The state vectors $\mathbf{x}^{(n)}$ will stabilize into the **steady-state (or stable) vector, s**. Which is to say, as $n \to \infty$, $\mathbf{x}^{(n)} \to \mathbf{s}$. If $\mathbf{s}$ exists:

$$T\mathbf{s} = \mathbf{s}$$

How to solve "s", the steady-state or stable vector:

Step 1: Set up the augmented matrix in the form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ & I-T & & 0 \end{array}\right)$$

(Of course, the number of 1's will depend on the size of our $(I-T)$ matrix.)

Step 2: Zero out whichever row you wish in the $(I-T)$ matrix by adding all the rows in the matrix together (do not include the row of 1's in your addition).*

Step 3: Get rid of any fractions or decimals using appropriate row operations.

Step 4: Continue to solve the system completely using Gauss-Jordan elimination, OR, ignoring the zero row, solve the system using Cramer's Rule.

Or, if you want an easier to remember approach to solve "s", the steady-state vector, simply do Step 1 above and proceed to row-reduce the augmented matrix into *RREF*. The solution to the system is "s". It can be a bit messy because fractions always show up, but you may prefer that to remembering all these fancy tricks I teach in this lesson.

Lecture Problems:

(Each of the questions below will be discussed and solved in the lecture that follows.)

1. Big Grocery Store loses 40% of its customers to Mom & Pop Groceries each month while Mom & Pop loses 50% of its customers to Big Grocery. Big Grocery Store currently has 70% of the market.

(a) Set up T, the transition matrix, for this problem using appropriate labels.

(b) Is this a regular Markov process?

(c) Find the market shares after the first month and after the second month.

(d) Find and interpret the stable vector for this Markov Chain.

(e) State the matrix that represents T^n as $n \to \infty$.

* Please note: in a 3×3 $(I - T)$ matrix, you will see me add three rows together to quickly zero out the last row (like question 7 part (d) in the *Practise Problems* below where I say $R_4 \to R_4 + R_3 + R_2$). Although this is totally correct algebraically, THIS IS NOT AN ELEMENTARY ROW OPERATION. (You are not allowed to add more than two rows together at once in elementary row operations.) Ask your prof if it is okay to do this to quickly zero out the row in these problems. If not, you could break it up into two steps that will be acceptable ($R_4 \to R_4 + R_3$ then $R_4 \to R_4 + R_2$) or just give up and row-reduce the way you always do and the row will zero itself out eventually.

2. Analysis has shown first-time voters are likely to vote the same way their parents have voted historically (Blue party or Red party). If their parents vote Blue party, a first-time voter is 70% likely to vote Blue also. If their parents vote Red party, a first-time voter is 80% likely to vote Red also. Assume a person votes for the same party for their entire life (ridiculous, I know), and the current generation of parents' votes are equally split between the two parties.

(a) Set up the transition matrix T and initial state vector S_0 for this Markov process. Label the rows and columns appropriately. Is this a regular Markov process?

(b) Compute S_1, the percent of first-time voters who will vote for each party in the next election.

(c) Compute S_2 and interpret it in this context.

(d) Find the stable vector S for this system and interpret it.

3. For many years, the only place to eat in town was Madge's Diner. Tomorrow, a fast food franchise, Burger Baron will open up. It is predicted, for each month thereafter, 2/7 of Madge's customers will switch over to the Burger Baron, while 1/7 of the Burger Baron's customers will switch over to Madge's Diner.

(a) Write the stochastic matrix using appropriate labels.

(b) Find the market share (fraction of all customers) for each restaurant one month and two months after Burger Baron opened.

(c) What fraction of customers will Madge's Diner retain in the long run?

4. Fresh Rite (FR), Klean Kwik (KK), and Super Press (SP) cleaners compete for the local dry cleaning market. Each year, analysts estimate FR will lose 10% of its customers to KK and 20% to SP, while KK will lose 30% to FR and 50% to SP, and SP will lose 60% to FR and 10% to KK. Suppose that FR currently has 40% of the market while the other two companies have an equal share.

(a) List the probability matrix using appropriate labels.

(b) Find the market shares of each company in one year's time.

(c) In the long run, what market share can each company expect?